

Zonal flow generation by parametric instability in magnetized plasmas and geostrophic fluids

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(Received 13 November 1999; accepted 8 February 2000)

Two-dimensional magnetized plasmas and geostrophic fluids exhibit a common nonlinearity due to the advection of vorticity. It is shown here that due to this nonlinearity, the propagation of small scale wave packets is accompanied by instability of a low frequency, long wavelength component. This instability is the coherent hydrodynamic generalization of the resonant type mean flow instability identified recently [P. H. Diamond, M. N. Rosenbluth, F. L. Hinton, M. Malkov, J. Fleischer, and A. Smolyakov, 17th IAEA Fusion Energy Conference, IAEA-CN-69/TH3/1, Yokohama, 1998 (to be published, International Atomic Energy Agency, Vienna)]. The mechanism discussed here, along with the resonant type, constitutes the ‘‘hydrodynamic’’ and ‘‘kinetic’’ regimes of the same process, similar to the case of plasma-beam instabilities. It is suggested that this generic mechanism is responsible for the generation of mean flow in atmospheres of rotating planets and magnetized plasmas. © 2000 American Institute of Physics. [S1070-664X(00)03605-3]

Development of anisotropic large scale structures, such as convective cells, zonal flows and jets, is a problem which has attracted a great deal of interest both in plasmas^{1,2} and in geophysical fluid dynamics [e.g., see Refs. 3 and 4 and references therein]. Recently it has been realized that zonal flows play a crucial role in the regulation of the anomalous transport in a tokamak.^{5–8} Scaling arguments show that, in general, transport of energy toward large scales is a result of the inverse energy cascade guaranteed in two-dimensional turbulence by the conservation of energy and enstrophy.⁹ A specific form of this mechanism can vary depending on conditions. For a number of physical situations, spontaneous excitation of large scale structures from small scale turbulence can be described as a negative eddy viscosity,^{10–14} so that the large scale perturbation with a wave number q will grow with a rate proportional to q^2 (or q^4 for other models).¹⁵ It has been shown^{8,16} that such an instability can be interpreted as a result of the resonant interaction between zonal flow and modulations of the small scale turbulence. In this letter we show that small scale wave packets can be subject to a more general instability leading to excitation of zonal flows. The latter instability manifests itself as a hydrodynamic, rather than kinetic, type interaction between the zonal flow and small scale fluctuations. Depending on a model, the instability has a larger growth rate which is proportional to $|q|$ or $|q|q^2$. This instability is somewhat similar to a parametric instability of a pump wave.^{2,17} The instability is generic to a wide variety of drift wave systems, such as plasma drift waves and Rossby waves in rotating fluids.

First, we consider drift waves in sheared magnetic field such as in a tokamak. We use a simple two-dimensional (in r, θ plane) model for electron drift waves¹⁸

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_0 \cdot \nabla\right) \frac{e\tilde{\phi}}{T_e} + \mathbf{V}_* \cdot \nabla \frac{e\tilde{\phi}}{T_e} - \rho_s^2 \left(\frac{\partial}{\partial t} + \mathbf{V}_0 \cdot \nabla + \tilde{\mathbf{V}}_E \cdot \nabla\right) \nabla_{\perp}^2 \frac{e\phi}{T_e} = 0. \quad (1)$$

Here, ρ_s^2 is the ion-sound Larmor radius and $\mathbf{V}_* = \hat{\theta}V_*$ is the electron diamagnetic drift velocity. The nonlinear equation (1) is similar to the Hasegawa-Mima model except for the term $\mathbf{V}_0 \cdot \nabla(e\tilde{\phi}/T_e)$, which is retained because the plasma density does not follow the Boltzmann distribution for large scale modes¹⁶ (a similar situation occurs in the sheared magnetic field for modes with $k_{\parallel} \rightarrow 0$ at the rational surface). The coupled dynamics of large scale flow and small scale turbulence are considered, so that the electrostatic potential ϕ is a sum of fluctuating $\tilde{\phi}$ and mean $\bar{\phi}$ quantities, $\mathbf{V}_0 = c\mathbf{b} \times \nabla \bar{\phi}/B_0$, $\tilde{\mathbf{V}}_E = c\mathbf{b} \times \nabla \tilde{\phi}/B_0$. The mean potential is the average of the total potential over fast, small scale variables and depends only on slow variables \mathbf{X} and T , $\bar{\phi} = \bar{\phi}(\mathbf{X}, T)$. The fluctuating part is a function of slow and fast, small scale (\mathbf{x}, t) variables $\tilde{\phi} = \tilde{\phi}(\mathbf{x}, t, \mathbf{X}, T)$. Averaging (1) over the fast, small scales, we obtain the evolution equation for the mean flow,

$$\frac{\partial}{\partial T} \nabla_{\perp}^2 \bar{\phi} = -\frac{c}{B_0} \overline{\{\tilde{\phi}, \nabla_{\perp}^2 \tilde{\phi}\}}, \quad (2)$$

which shows that the large scale flow is driven by small scale fluctuations via Reynolds stress forces. Here, $\{a, b\} = \partial_r a \partial_{\theta} b - \partial_{\theta} a \partial_r b$ is the Poisson bracket, and B_0 is the equilibrium magnetic field.

Coupling of small scale fluctuations to the mean flow is described by the kinetic equation for wave packets¹⁹

$$\frac{\partial N_k}{\partial T} + \frac{\partial \omega_k}{\partial \mathbf{k}} \cdot \frac{\partial N_k}{\partial \mathbf{X}} - \frac{\partial \omega_k}{\partial \mathbf{X}} \cdot \frac{\partial N_k}{\partial \mathbf{k}} = S, \quad (3)$$

where $N_k = N_k(\mathbf{X}, T)$ is the adiabatic action invariant, and the exact form of N_k is model dependent. For the model given by Eq. (1), the wave frequency is $\omega_k = k_\theta V_0 + \omega_k^l$, where $\omega_k^l = k_\theta V_* / (1 + k_\perp^2 \rho_s^2)$ is the local wave frequency, and the mean flow \mathbf{V}_0 enters the total frequency ω_k as a simple Doppler shift. The drift wave + zonal flows system described by Eq. (1) has the adiabatic action invariant^{18,20}

$$N_k = \mathcal{E} / \omega_k^l = (1 + k_\perp^2 \rho_s^2)^2 \left| \frac{e \phi_k}{T_e} \right|^2, \quad (4)$$

where \mathcal{E} is the wave energy, $\rho_s^2 = T_e / m_i \omega_{ci}^2$.

The source term in (3) describes the wave growth and damping due to linear and nonlinear mechanisms. Symbolically, one can represent it as $S = \gamma_k N_k - \Delta \omega_k N_k^2$, where γ_k is the linear growth rate, and $\Delta \omega_k N_k^2$ is the damping term due to nonlinear broadening effects. We assume that small scale turbulence is close to a stationary state, so that $S \rightarrow 0$.

Coupled equations (2) and (3) can be solved to show that the modulations of the wave packets and zonal flow \mathbf{V}_0 are unstable.⁸ We consider equations (2) and (3) linearized for small perturbations $(\tilde{N}_k, \bar{\phi}) \sim \exp(-i\Omega T + iqr)$, where $q \equiv q_r = -i \partial / \partial r$ is the radial wave vector of the large scale perturbation. Then, Eq. (2) takes the form

$$\frac{\partial}{\partial T} \nabla_\perp^2 \bar{\phi} = \frac{c}{B_0} \nabla_r^2 (\nabla_r \bar{\phi} \nabla_\theta \phi), \quad (5)$$

and

$$-i\Omega \bar{\phi} = \frac{c}{B_0} \int k_r k_\theta |\phi_k|^2 d^2 k. \quad (6)$$

The modulation of \tilde{N}_k is calculated from (3),

$$\tilde{N}_k = -\frac{c}{B_0} q^2 \bar{\phi} k_\theta \frac{\partial N_k^0}{\partial k_r} \frac{i}{\Omega - qV_g}, \quad (7)$$

where $V_g = \partial \omega / \partial k_r$. Using (7) in (6) we obtain the following equation⁸:

$$-i\Omega = -q^2 c^2 \int d^2 k \frac{k_\theta^2 \rho_s^2}{(1 + k_\perp^2 \rho_s^2)^2} k_r \frac{\partial N_k^0}{\partial k_r} \frac{i}{\Omega - qV_g}. \quad (8)$$

The resonant type instability is obtained from (7) by using the resonant function $R = i / (\Omega - qV_g) \rightarrow \pi \delta(\Omega - qV_g)$ or its broadened counterpart $i / (\Omega - qV_g + \Delta \omega_k)$ (for a white noise source, this can be taken as $1 / \Delta \omega_k$, where $\Delta \omega_k$ is nonlinear broadening due to the wave-wave interaction). This instability may be interpreted as a result of the resonant interaction of the wave packet with slow modulations of the mean flow. For the case of the narrow resonant function approximated by a delta function, the growth rate of the resonant instability is

$$\gamma_q = -q^2 c^2 \int d^2 k \frac{k_\theta^2 \rho_s^2}{(1 + k_\perp^2 \rho_s^2)^2} k_r \frac{\partial N_k^0}{\partial k_r} \pi \delta(\Omega - qV_g). \quad (9)$$

The condition $\partial N_k^0 / \partial k_r < 0$ is required for instability.

In this letter we show that Eq. (8) also describes another type of the instability that is not of the resonant type, but rather of the hydrodynamic variety. When the growth rate of the instability becomes large compared to the characteristic frequency spread for the background fluctuations, individual N_k components contribute to the instability coherently. Insight into this mechanism can be provided by a simple case of a monochromatic wave packet with $N_k^0 = N_0 \delta(\mathbf{k} - \mathbf{k}_0)$, with $\mathbf{k}_0 = (k_{r0}, k_{\theta 0})$.

Performing integration by parts in (8) we reduce it to

$$-\Omega = q^2 c^2 \int d^2 k k_\theta^2 \rho_s^2 N_k^0 \frac{\partial}{\partial k_r} \left(\frac{1}{\Omega - qV_g} \frac{k_r}{(1 + k_\perp^2 \rho_s^2)^2} \right). \quad (10)$$

Noting that for drift waves

$$\frac{k_r}{(1 + k_\perp^2 \rho_s^2)^2} = -\frac{V}{2k_\theta V_* \rho_s^2}, \quad (11)$$

we rewrite equation (10) in the form

$$1 = q^2 c^2 \int d^2 k k_\theta^2 \frac{N_k^0}{2k_\theta V_*} \frac{\partial V_g}{\partial k_r} \frac{1}{(\Omega - qV_{gr})^2}, \quad (12)$$

or

$$(\Omega - qV_{gr})^2 = q^2 c^2 k_\theta^2 \frac{N_k^0}{2k_\theta V_*} \frac{\partial V_g}{\partial k_r}. \quad (13)$$

Note that the criterion for the instability is thus

$$\frac{N_k^0}{2k_\theta V_*} \frac{\partial V_g}{\partial k_r} < 0. \quad (14)$$

Calculating the derivative of the group velocity we obtain

$$\Omega = qV_{gr} - i|q|c_s \frac{|k_{\theta 0} \rho_s|}{(1 + k_\perp^2 \rho_s^2)^{3/2}} N_0^{1/2} \sqrt{1 - 4k_{r0}^2 \rho_s^2 + k_\perp^2 \rho_s^2}. \quad (15)$$

This equation describes a growth of the large scale zonal flow as a result of the instability. Note that the instability is stabilized for shorter wavelengths, provided that $1 - 3k_{r0}^2 \rho_s^2 + k_\perp^2 \rho_s^2 < 0$. It can readily be seen that the coherent (hydrodynamic) instability has a larger growth rate compared to that of the resonant instability (9).

We have considered a specific example of drift waves in plasmas, but, similar arguments can be made for Rossby-type waves in fluids. For the systems of interest (magnetized plasma and geostrophic fluids of rotating planets), the conservation of potential vorticity is an essential characteristic of wave dynamics. In all cases, nonlinear advection of the potential vorticity remains a source of large scale motion, though the exact form for the potential vorticity conservation for different types of waves in plasma and rotating fluids may vary. One of the most general forms for the vorticity conservation is the Hasegawa-Mima or Charney-Obukhov equation

$$\partial_t (\psi - \nabla_\perp^2 \psi) + \partial_\theta \psi - \{\psi, \nabla_\perp^2 \psi\} = 0, \quad (16)$$

where the details of various normalizations for different types of plasma and Rossby waves are given in Refs. 21–23 and references therein. Here ψ is the stream-function for two-

dimensional velocity in the θ, r -plane (β -plane), and is as a sum of the mean flow and small scale fluctuations, $\psi = \bar{\psi} + \tilde{\psi}$. The system given by Eq. (16) has an adiabatic invariant^{13,24,20,25} $N_k = k_{\perp}^2(1 + k_{\perp}^2)|\psi_k|^2$ and the wave frequency $\omega_k = k_{\theta}/(1 + k_{\perp}^2) + k_{\theta}V_0k_{\perp}^2/(1 + k_{\perp}^2)$. Then Eqs. (6) and (7) are replaced with the following equations:

$$-i\Omega(1 + q^2)\bar{\psi} = q^2 \int k_r k_{\theta} |\psi_k|^2 d^2k, \quad (17)$$

$$\tilde{N}_k = -q^2 \bar{\psi} k_{\theta} \frac{k_{\perp}^2}{1 + k_{\perp}^2} \frac{\partial N_k^0}{\partial k_r} \frac{i}{\Omega - qV_g}. \quad (18)$$

Solving these equations, instead of (13), we obtain

$$(\Omega - qV_{gr})^2 = \frac{q^4}{1 + q^2} k_{\theta}^2 \frac{N_k^0}{2k_{\theta}} \frac{\partial V_{gr}}{\partial k_r}. \quad (19)$$

It is interesting to note that despite different definitions of the wave action and different contribution of the mean flow to the eigenfrequency, the criterion for the instability (14) remains the same. It is interesting to note that in the model for drift waves given by Eq. (1), zonal flow dynamics is similar to that of two-dimensional Euler fluid. In other models, such as the Hasegawa-Mima equation and the Charney-Obukhov equation for Rossby waves, the equation for a driven zonal flow has different structure [see Eq. (17)], and, as a result, zonal flow should have different spectral distribution.

As follows from the above analysis, the small scale wave packets in magnetized plasmas and geostrophic fluids are unstable with respect to the long wavelength perturbations. These perturbations are accompanied by the excitation of the long wavelength modes of the velocity, i.e., zonal flows. This robust instability is the coherent hydrodynamic generalization of the resonant type mean flow instability recently considered.⁸ This mechanism, along with the resonant type, constitutes the ‘‘hydrodynamic’’ and ‘‘kinetic’’ regimes of the same process, similar to the case of plasma-beam instabilities. Relative importance of these two regimes will be determined by the relation between the nonlinear growth rate γ_q [given by either Eq. (9) or (15)] and the spectral width of the background turbulence, $\delta\omega_k$. The instability is of the resonant type, when the instability growth rate γ_q is smaller than the spectral width $\delta\omega_k$ of the small scale fluctuations. The instability becomes the coherent hydrodynamic type if $\gamma_q > \delta\omega_k$, so that all harmonics grow coherently. In the simplest case of weak wave-wave interaction, the spectral width $\delta\omega_k$ is merely the width of the wave packet of small scale fluctuations. The finite wave-wave interaction will further broaden the spectrum and the nonlinear broadening $\Delta\omega_k$ must be taken into account in the estimate for the spectrum width.

We suggest that the generic mechanism discussed in this paper is responsible for generation of mean flow in atmospheres of rotating planets and magnetized plasmas. It is interesting to note that the criterion for the instability given by Eq. (14) is somewhat similar to a general Lighthill criterion

for modulational instability.²⁷ Note that the existence of the eigenmodes in the wave dynamics is essential for the discussed instability mechanism. We would like to note that large scale perturbations with a nonzero longitudinal number q_{θ} could also be unstable.¹³ The instability with $q_{\theta} \neq 0$ and $q_r = 0$ corresponds to a more conventional modulational instability of one-dimensional wave packets (i.e., the large scale instability develops in the direction of the wave propagation of the small scale packet²⁸) (see also Ref. 29 and references therein). The discussed mechanism of the zonal flow instability can be extended to the general case $q_{\theta} \neq 0$; however, such theory is somewhat more complicated and beyond the scope of the present letter. (A similar mechanism was also considered in Ref. 26).

ACKNOWLEDGMENTS

This research was supported by the Natural Sciences and Engineering Research Council of Canada and U.S. Department of Energy Grant No. FG03-88ER53275.

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